Annihilating Polynomials of Quadratic Forms

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Abstract

The focus of this talk lies on annihilating polynomials and the annihilating ideal of a single, given quadratic form rather than on polynomials annihilating the Witt ring, its fundamental ideal, its torsion part or any other class of quadratic forms.

We start by giving a short introduction to the algebraic theory of quadratic forms and the theory of Witt rings. Next we define annihilating polynomials and give a method to calculate them. After examining the general anatomy of annihilating polynomials we then study the example of quadratic forms over local fields and by means of this example point out some open questions.

1 Witt Rings

We define the natural numbers $\mathbb{N}$ not to contain 0. If we need 0 as a natural number we use the set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

During all of this talk K will be a field with $\mathrm{char}(K) \neq 2$.

1.1 Definition. *A* quadratic form *of dimension $n \in \mathbb{N}_0$ over K is a homogeneous polynomial $\varphi \in K[X_1, \dots, X_n]$ of degree 2. We use the notation* $\dim(\varphi) = n$.

1.2 Remark. The dimension is an integral part of the definition of a quadratic form. For example even $X_1^2 \in K[X_1, X_2, X_3]$ can be considered as a 3-dimensional quadratic form over K. It is necessary to define quadratic forms like that in order to avoid conflicts with the geometric theory of quadratic forms. △

An n-dimensional quadratic form φ over K defines a *quadratic map*

$$\varphi : \quad K^n \longrightarrow K, \quad (x_1, \dots, x_n)^t \longmapsto \varphi(x_1, \dots, x_n),$$

which induces a *symmetric bilinear form*

$$b_\varphi : \quad K^n \times K^n \longrightarrow K, \quad (v, w) \longmapsto \frac{1}{2}(\varphi(v+w) - \varphi(v) - \varphi(w)),$$

which in its turn defines a symmetric matrix

$$A_\varphi \ := \ (b_\varphi(e_i, e_j))_{i,j=1,\dots,n} \ \in \ \mathbb{M}_n(K),$$

where $\{e_1, \ldots, e_n\}$ denotes the standard basis of K^n.

We have

$$b_\varphi(v, w) \;=\; v^t A_\varphi w$$

and

$$\varphi(v) \;=\; b_\varphi(v, v)$$

for $v, w \in K^n$.

1.3 Definition. *Two n-dimensional quadratic forms φ and ψ over K are* isometric *if there exists a linear transformation $T \in \mathrm{GL}_n(K)$ such that $\varphi(v) = \psi(Tv)$ for all $v \in K^n$. We write $\varphi \cong \psi$.*

It is well-known that every n-dimensional quadratic form φ over K is isometric to a *diagonal form*, i.e. a quadratic form ψ whose associated matrix A_ψ is a diagonal matrix (see [Lam05, Chapter 1, Corllary 2.4]). If $a_1, \ldots, a_n \in K$ are the entries of A_ψ, then we write

$$\psi \;=\; \langle a_1, \ldots, a_n \rangle$$

or $\varphi \cong \langle a_1, \ldots, a_n \rangle$.

1.4 Definition. *A quadratic form φ over K is called* non-degenerate *or* regular *if* $\det(A_\varphi) \neq 0$.

1.5 Remark. It is clear that φ is non-degenerate if and only if we have $\varphi \cong \langle a_1, \ldots, a_n \rangle$ with some $a_1, \ldots, a_n \neq 0$. △

1.6 Remark. There are many different ways to define quadratic forms. Some authors prefer to talk about quadratic maps and use the notion "quadratic form" for non-degenerate quadratic maps. △

Since we are only interested in non-degenerate quadratic forms, henceforth a *form* will be a non-degenerate quadratic form.

Let φ and ψ be forms over K, $\dim(\varphi) = n$ and $\dim(\psi) = m$. We define the *orthogonal sum*

$$\varphi \perp \psi : \quad K^n \oplus K^m \longrightarrow K, \quad v + w \longmapsto \varphi(v) + \psi(w),$$

and the *tensor product*

$$\varphi \otimes \psi : \quad K^n \otimes K^m \longrightarrow K, \quad v \otimes w \longmapsto \varphi(v)\psi(w).$$

Obviously we have $\dim(\varphi \perp \psi) = \dim(\varphi) + \dim(\psi)$ and $\dim(\varphi \otimes \psi) = \dim(\varphi)\dim(\psi)$.

Consider the set $\widehat{W}^+(K)$ consisting of isometry classes of forms over K. For a form φ we denote by $[\varphi]$ its isometry class. If $\varphi \cong \langle a_1, \ldots, a_n \rangle$ and $\psi \cong \langle b_1, \ldots, b_m \rangle$ are forms over K, then we can define

$$[\varphi] + [\psi] \;:=\; [\varphi \perp \psi] \;=\; [\langle a_1, \ldots, a_n, b_1, \ldots, b_m \rangle]$$

and

$$[\varphi] \cdot [\psi] \;:=\; [\varphi \otimes \psi] \;=\; [\langle a_1 b_1, \ldots, a_1 b_m, a_2 b_1, \ldots, a_n b_m \rangle]\,,$$

which makes $\widehat{W}^+(K)$ a commutative semi-ring.

Now we can apply the Grothendieck construction to $\widehat{W}^+(K)$ to obtain the *Witt-Grothendieck ring* $\widehat{W}(K)$. This construction associates to a semi-group (resp. semi-ring) a group (resp. ring) in an analogous fashion as one obtains the integers from the natural numbers (compare [Sch85, Chapter 2, Theorem 1.1]). More specifically, the elements of $\widehat{W}(K)$ are formal differences

$$x \;=\; [\varphi] - [\psi]\,, \quad [\varphi]\,,[\psi] \in \widehat{W}^+(K).$$

Here we have $x = 0$ if and only if $\varphi \cong \psi$. The unit element of $\widehat{W}(K)$ is given by $[\langle 1\rangle]$.

Before we can now define the Witt ring of K, we first need to introduce some definitions, notation and results.

1.7 Definition. *Let φ be an n-dimensional form over K.*

(1) A form ψ over K is called a subform *of φ if there exists a form χ over K with $\varphi \cong \psi \perp \chi$.*

(2) The form φ represents *an element $a \in K$ if there exists a $v \in K^n$ with $v \neq 0$ and $\varphi(v) = a$.*

(3) We call φ isotropic *if φ represents 0. Otherwise φ is called* anisotropic.

1.8 Remark. Let K be a field.

(1) It is clear that every 1-dimensional form over K is anisotropic.

(2) Up to isometry there exists only one 2-dimensional isotropic form over K, i.e. the *hyperbolic plane* $\mathbb{H} = XY \in K[X,Y]$. We have $\mathbb{H} \cong \langle a, -a\rangle$ for all $a \in K^*$ (see [Sch85, Chapter 1, Corollary 4.6]).

$\triangle$

For a quadratic form φ over K and an $m \in \mathbb{N}_0$ we define

$$m \times \varphi \;:=\; \underbrace{\varphi \perp \ldots \perp \varphi}_{m\text{-times}}.$$

1.9 Theorem. (Witt's Decomposition Theorem)
Every non-degenerate quadratic form φ over K can be decomposed as follows:

$$\varphi \;\cong\; \varphi_{\mathrm{an}} \perp i(\varphi) \times \mathbb{H},$$

where $i(\varphi) \in \mathbb{N}_0$ is uniquely determined, and φ_{an} is anisotropic and uniquely determined up to isometry.

[Lam05, Chapter 1, Theorem 4.1]

1.10 Definition. *For a form φ over K the subform φ_{an} from the previous theorem is called* anisotropic kernel *of φ. The natural number $i(\varphi)$ is called the* Witt index *of φ.*

1.11 Definition. *A form φ over K is called* hyperbolic *if φ is isometric to an orthogonal sum of hyperbolic planes, or in other words if there exists an $i \in \mathbb{N}_0$ such that $\varphi \cong i \times \mathbb{H}$.*

Let φ be a form over K, $\varphi = \langle a_1, \ldots, a_n\rangle$, and let $b \in K^*$. We set

$$b\varphi \;:=\; \langle ba_1, \ldots, ba_n\rangle.$$

Note that in general $-[\varphi] \neq [-\varphi]$ in $\widehat{W}(K)$.

Now we can define an equivalence relation on the class of non-degenerate quadratic forms over K by setting

$$\varphi \sim \psi \quad :\Longleftrightarrow \quad \varphi_{\mathrm{an}} \cong \psi_{\mathrm{an}}.$$

It is easy to check that this is indeed an equivalence relation. The equivalence class of a form φ over K is denoted by $\{\varphi\}$. Let $W(K)$ be the set of equivalence classes of forms over K.

The orthogonal sum and the tensor product induce an addition

$$\{\varphi\} + \{\psi\} \ := \ \{\varphi \perp \psi\}$$

and a multiplication

$$\{\varphi\} \cdot \{\psi\} \ := \ \{\varphi \perp \psi\}$$

on $W(K)$. Since $\varphi \perp -\varphi$ is hyperbolic, it follows that $-\{\varphi\} = \{-\varphi\}$. This shows that $W(K)$ is a commutative ring with unit element $\{\langle 1 \rangle\}$. We call $W(K)$ the *Witt ring* of K.

1.12 Remark. From the construction of $W(K)$ it becomes clear, that $W(K)$ can be used to classify anisotropic forms over K. △

1.13 Remark. It is also possible to obtain $W(K)$ as a quotient of $\widehat{W}(K)$. If $J \subset \widehat{W}(K)$ is the ideal generated by $[\mathbb{H}]$, then $W(K) \cong \widehat{W}(K)/J$. △

2 Annihilating Polynomials

2.1 Definition. *Let R be a commutative, unitary ring, and let $\iota : \mathbb{Z} \to R$ be the canonical homomorphism. A polynomial $P = a_n X^n + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$ is called* annihilating polynomial *of an element $x \in R$ if*

$$P(x) \ := \ \iota(a_n)x^n + \ldots \iota(a_1)x + \iota(a_0) \ = \ 0.$$

Usually we will simply omit the ι and write $a_i x^i$ instead of $\iota(a_i)x^i$.

The study of annihilating polynomials of quadratic forms was initiated by David W. Lewis in 1987 when he introduced for $n \in \mathbb{N}_0$ the polynomial

$$P_n \ := \ (X - n)(X - n + 2) \cdots (X + n - 2)(X + n) \ \in \ \mathbb{Z}[X] \tag{1}$$

and proved that it annihilates the isometry class of every n-dimensional quadratic form over any field K (see [Lew87]). Rather than reproducing Lewis' approach here, we follow an approach introduced by Jürgen Hurrelbrink in [Hur89].

Let $G := K^*/(K^*)^2$ be the *square class group* of K, and for $a \in K^*$ denote by $\overline{a}$ the image of a in G. There exists a natural ring homomorphism

$$\pi_1 : \quad \mathbb{Z}[G] \longrightarrow \widehat{W}(K)$$

defined by

$$\overline{a_1} + \cdots + \overline{a_n} \longmapsto [\langle a_1, \ldots, a_n \rangle].$$

If $\pi : \widehat{W}(K) \to W(K) = \widehat{W}(K)/_J$ is the canonical projection, then we set

$$\pi_2 := \pi \circ \pi_1 : \quad \mathbb{Z}[G] \longrightarrow W(K).$$

We will now proceed by constructing annihilating polynomials for elements of $\mathbb{Z}[G]$. Via π_1 and π_2 we thus obtain annihilating polynomials for elements of $\widehat{W}(K)$ and $W(K)$.

Consider the dual $\mathrm{Hom}(\mathbb{Z}[G], \mathbb{Z})$. It follows from [Hur89, Lemma 1.1] that

$$\bigcap_{\chi \in \mathrm{Hom}(\mathbb{Z}[G],\mathbb{Z})} \ker(\chi) \;=\; \{0\}. \tag{2}$$

For $f \in \mathbb{Z}[G]$ we now set

$$S_f \;:=\; \{\chi(f) \mid \chi \in \mathrm{Hom}(\mathbb{Z}[G], \mathbb{Z})\}\,.$$

Note that for $\chi \in \mathrm{Hom}(\mathbb{Z}[G], \mathbb{Z})$ we must have $\chi(g) = \pm 1$ for all $g \in G$, since $g^2 = 1$. It follows that if $f = \sum_{g\in G} a_g g$ and $b = \sum_{g\in G} |a_g|$, then $-b \le \chi(f) \le b$ for all $\chi \in \mathrm{Hom}(\mathbb{Z}[G], \mathbb{Z})$. This shows that S_f is finite. Hence we can define the polynomial

$$P_f \;:=\; \prod_{\chi(f)\in S_f} (X - \chi(f)) \;\in\; \mathbb{Z}[G].$$

Since $\chi(P_f(f)) = 0$ for all $\chi \in \mathrm{Hom}(\mathbb{Z}[G], \mathbb{Z})$, it follows from (2) that $P_f(f) = 0$, i.e. P_f is an annihilating polynomial of f.

2.2 Definition. *An element $f \in \mathbb{Z}[G]$ is called a* preform *of dimension $n \in \mathbb{N}_0$, if $f = \overline{a_1} + \cdots + \overline{a_n}$ with $a_1, \ldots, a_n \in K^*$. We also say that f is a preform of $\langle a_1, \ldots, a_n\rangle$.*

Now if f is a preform of dimension n. Then it is clear that

$$-n \le \chi(f) \le n \qquad \text{and} \qquad \chi(f) \equiv n \pmod 2$$

for all $\chi \in \mathrm{Hom}(\mathbb{Z}[G], \mathbb{Z})$. This implies that the Lewis polynomial

$$P_n \;=\; (X - n)(X - n + 2)\cdots(X + n - 2)(X + n)$$

as defined in (1) annihilates f. We conclude that P_n annihilates the isometry class of every n-dimensional quadratic form over K.

2.3 Remark. It is easy to show that the set of all annihilating polynomials of an element $f \in \mathbb{Z}[G]$ is exactly the principal ideal (P_f). For the Witt-Grothendieck and the Witt ring the situation is usually a lot more complicated as we will see. △

3 The Embracing Polynomial

Let K be a field with $\mathrm{char}(K) \neq 2$, and let $G := K^*/(K^*)^2$. In this section we denote by R either $\widehat{W}(K)$ or $W(K)$

3.1 Proposition. *For $x \in R$ there exists a unique monic polynomial $Q_x \in \mathbb{Z}[X]$ such that* *(i) Q_x divides all annihilating polynomials of x and*

(ii) Q_x is of maximal degree among all monic polynomials in $\mathbb{Z}[X]$ that satisfy property (i).

Furthermore Q_x is a product of linear factors, and $Q_x(x)$ is a torsion element of R.

Proof. Let $f \in \mathbb{Z}[G]$ be a preform of a representative for the class x. We have seen above that the polynomial P_f annihilates f and therefore also x.

Consider an annihilating polynomial $P \neq 0$ of x such that P has minimal degree. Set $Q_x := \gcd(P, P_f)$. Then there exist $\lambda, \mu \in \mathbb{Z}[X]$ and an $m \in \mathbb{N}$ such that $mQ_x = \lambda P + \mu P_f$. This implies $(mQ_x)(x) = 0$, i.e. mQ_x is an annihilating polynomial of x and $Q_x(x)$ lies in R_t. Since by our assumption P has minimal degree, we must have $\deg(Q_x) = \deg(P)$. Furthermore since P_f is a product of linear factors, the same must be true for Q_x.

It remains to show that Q_x divides every annihilating polynomial P' of x. But this can easily be shown by considering $\gcd(mQ_x, P')$, an integer multiple of which must be an annihilating polynomial. Since mQ_x is of minimal degree, it follows that $\gcd(mQ_x, P')$ and Q_x must have the same degree. Thus Q_x divides P'. □

3.2 Definition. *Let $x \in R$.*

(1) The polynomial Q_x from the previous proposition is called embracing polynomial *of x.*

(2) The annihilating ideal *of x is the ideal $\mathrm{Ann}_x \subset \mathbb{Z}[X]$ consisting of all annihilating polynomials of x.*

3.3 Remark. The specific name "embracing polynomial" was chosen in view of the fact that $\mathrm{Ann}_x \subset (Q_x)$ for $x \in R$, and (Q_x) is the unique minimal principal ideal that contains Ann_x. △

Next we will introduce a method that allows calculating the embracing polynomial.

Consider the set of ring homomorphisms $\mathrm{Hom}(R, \mathbb{Z})$ and for $x \in R$ define

$$S_x^{\mathrm{sign}} := \{\chi(x) \mid \chi \in \mathrm{Hom}(R, \mathbb{Z})\}.$$

Since every homomorphism $R \to \mathbb{Z}$ is induced by a homomorphism $\mathbb{Z}[G] \to \mathbb{Z}$, it follows from the above that S_x^{sign} is finite. Therefore we can define the *signature polynomial*

$$P_x^{\mathrm{sign}} := \prod_{\chi(x) \in S_x^{\mathrm{sign}}} (X - \chi(x)) \in \mathbb{Z}[X].$$

If $\mathrm{Hom}(R, \mathbb{Z}) = \varnothing$ then P_x^{sign} is the empty product and hence equal to 1.

Consider the intersection $\bigcap_{\chi \in \mathrm{Hom}(R,\mathbb{Z})} \ker(\chi)$. If $\mathrm{Hom}(R, \mathbb{Z}) = \varnothing$, then we define the empty intersection to be all of R. It is well-known (see [Sch85, Chapter 2, Theorem 7.3]) that in all cases

$$R_t = \bigcap_{\chi \in \mathrm{Hom}(R,\mathbb{Z})} \ker(\chi).$$

In other words $x \in R$ is torsion if and only if $\chi(x) = 0$ for all $\chi \in \mathrm{Hom}(R, \mathbb{Z})$.

3.4 Proposition. *If $x \in R$, then the embracing polynomial Q_x is equal to the signature polynomial P_x^{sign}.*

Proof. As $\chi(P_x^{\mathrm{sign}}(x)) = P_x^{\mathrm{sign}}(\chi(x)) = 0$ for all $\chi \in \mathrm{Hom}(R, \mathbb{Z})$, we deduce that $P_x^{\mathrm{sign}}(x)$ is torsion, i.e. there exists some $m \in \mathbb{N}$ such that mP_x^{sign} is an annihilating polynomial of x. Hence by definition Q_x divides mP_x^{sign}, and since Q_x is monic it must also divide P_x^{sign}. But now Q_x is a product of linear factors and $\chi(Q_x(x)) = Q_x(\chi(x)) = 0$ for all $\chi \in \mathrm{Hom}(R, \mathbb{Z})$. Therefore, since $\mathbb{Z}[X]$ is factorial, $\chi(x)$ is a root of Q_x or equivalently $X - \chi(x)$ divides Q_x for all $\chi \in \mathrm{Hom}(R, \mathbb{Z})$. Altogether we obtain that P_x^{sign} divides Q_x and hence $Q_x = P_x^{\mathrm{sign}}$. □

3.5 Remark. There exists a more natural setting for the study of annihilating polynomials of quadratic forms. If G is any group of exponent 2, and if $R = \mathbb{Z}[G]/J$ such that R has only 2-torsion, then R is called a *Witt ring for G*. In [KRW72] Knebusch, Rosenberg and Ware study the properties of these rings. They discover that those rings share a large number of useful properties with the Witt-Grothendieck ring and the Witt ring. These two rings themselves are Witt rings for the square class group of a field. It is possible to generalize all of the above results about annihilating polynomials to Witt rings for groups of exponent 2. △

Recall that a field K is called formally real if -1 cannot be written as a sum of squares. Otherwise there exists a minimal $s \in \mathbb{N}$ such that -1 can be written as a sum of s squares. This s is called the *level* of the field K. We denote it by $s(K)$. Pfister discovered that $s(K)$ is always a power of 2 (see [Pfi95, Chapter 3.1]). It is well-known that over a field K with $s(K) = 2^l$ we have $2^{l+1} \times \langle 1 \rangle \sim 0$. We set $s(K) := +\infty$ in the case that K is formally real.

For every field K there exists the dimension homomorphism $\dim : \widehat{W}(K) \to \mathbb{Z}$. If K is formally real, then there exists at least one signature homomorphism $W(K) \to \mathbb{Z}$. If K is not formally real, then there does not exist a ring homomorphism $W(K) \to \mathbb{Z}$. Thus in that case it follows from the above that $W(K)_t = W(K)$.

Let φ be a quadratic form of dimension n over a field K. First we assume that K is not formally real. In this case it is easy to explicitly give the embracing polynomial of both $[\varphi] \in \widehat{W}(K)$ and $\{\varphi\} \in W(K)$. Recall that the dimension homomorphism $\dim : \widehat{W}(K) \to \mathbb{Z}$ is the only ring homomorphism $\widehat{W}(K) \to \mathbb{Z}$. Hence

$$Q_{[\varphi]} \;=\; X - \dim(\varphi) \;=\; X - n \;\in\; \mathbb{Z}[X].$$

For the Witt Ring we have $\mathrm{Hom}(W(K), \mathbb{Z}) = \varnothing$, which implies

$$Q_{\{\varphi\}} \;=\; 1.$$

Indeed 1 divides every annihilating polynomial of $\{\varphi\}$, and furthermore, since $W(K)$ is a torsion ring, the element $1 \in W(K)$ is torsion.

Now let K be formally real. In this case $\mathrm{Hom}(\widehat{W}(K), \mathbb{Z})$ consists of the signature homomorphisms and the dimension homomorphism whereas $\mathrm{Hom}(W(K), \mathbb{Z})$ only consists of the signature homomorphisms. More specifically there exists a bijection

$$\mathrm{Hom}(W(K), \mathbb{Z}) \longrightarrow \mathrm{Hom}(\widehat{W}(K), \mathbb{Z}) \setminus \{\dim\}, \quad \chi \longmapsto \chi \circ \pi,$$

where $\pi : \widehat{W}(K) \to W(K)$ is the canonical projection. In this situation we thus obtain

$$Q_{[\varphi]} \;=\; (X - n) Q_{\{\varphi\}}.$$

4 Applications and Open Questions

We first consider an example, namely we will give full sets of generators for the annihilating ideal of an arbitrary form over a local field. So let K be a local field with finite residue field.

Let $I(K) \subset W(K)$ be the ideal consisting of the equivalence classes of all even-dimensional forms over K. This ideal is called the *fundamental ideal* of $W(K)$. The powers of the fundamental ideal play an important role in the algebraic theory of quadratic forms. It is well-known that over local fields the third power of $I(K)$ vanishes, and hence it is possible to classify quadratic forms over K completely with the help of the first three invariants:

- The *dimension index*:

$$e_0: \quad W(K) \longrightarrow \mathbb{Z}/2\mathbb{Z}, \quad \{\varphi\} \longmapsto \dim(\varphi) \pmod 2.$$

- The *discriminant*:

$$d: \quad W(K) \longrightarrow K^*/(K^*)^2, \quad \{\varphi\} \longmapsto (-1)^{\frac{n(n-1)}{2}} \det(\varphi) \pmod{(K^*)^2},$$

  where $n = \dim(\varphi)$ and $\det(\varphi) = \det(A_\varphi)$.

- The *Clifford invariant*:

$$c: \quad W(K) \longrightarrow {}_2\mathrm{Br}(K).$$

These invariants can now be used to determine a full set of generators for the annihilating ideal of a quadratic form over K. We will omit the calculations here.

Let φ be an n-dimensional quadratic form over K. First we consider annihilating polynomials of $[\varphi]$. Since the third power of the fundamental ideal vanishes we must also have $8 \times \langle 1 \rangle \sim 0$, which implies that K is not formally real. In fact only the cases $s(K) = 1$ and $s(K) = 2$ are possible. Hence we obtain $Q_{[\varphi]} = X - n \in \mathbb{Z}[X]$. The case $\varphi \cong n \times \langle 1 \rangle$ is trivial since

$$\mathrm{Ann}_{[\varphi]} \;=\; (X - n).$$

If $\varphi \not\cong n \times \langle 1 \rangle$ then

$$\mathrm{Ann}_x \;=\; \begin{cases} (2(X-n), (X-r)(X-n)) & \text{if } \det(\varphi) \text{ is a sum of two squares in } K, \\ (4(X-n), (X-r)(X-n)) & \text{otherwise,} \end{cases}$$

for some $r \in \mathbb{Z}$.

Now we consider annihilating polynomials of $\{\varphi\}$. Since K is not formally real, we have $Q_{\{\varphi\}} = 1$. If $\varphi \sim r \times \langle 1 \rangle$ for some $r \in \mathbb{Z}$, then obviously

$$\mathrm{Ann}_{\{\varphi\}} \;=\; (2s(K), X - r).$$

Assume now that $\varphi \not\sim r \times \langle 1 \rangle$ for all $r \in \mathbb{Z}$. If $s(K) = 1$, then

$$\mathrm{Ann}_{\{\varphi\}} \;=\; \begin{cases} (2, X^2) & \text{for } n \text{ even,} \\ (2, (X+1)^2) & \text{for } n \text{ odd.} \end{cases}$$

If $s(K) = 2$ and n is even, then

$$\mathrm{Ann}_{\{\varphi\}} = \begin{cases} (4, 2X, X^2) & \text{if } d(\varphi) \text{ is a sum of two squares in } K, \\ (4, X(X+2)) & \text{otherwise.} \end{cases}$$

In the case $s(K) = 2$ and n odd we obtain

$$\mathrm{Ann}_{\{\varphi\}} = \begin{cases} (4, 2(X+1), (X+1)^2) & \text{if } -d(\varphi) \text{ is a sum of two squares in } K, \\ (4, (X-1)(X+1)) & \text{otherwise.} \end{cases}$$

Via the Hasse-Minkowsky theorem these results about local fields can now be applied to determine full sets of generators for the annihilating ideals of forms over global fields.

The just treated example already serves well to point out a few open questions and problems.

Let K be an arbitrary field, let φ be a form over K, and let x denote the class of φ in R which can be either $\widehat{W}(K)$ or $W(K)$. There exist monic polynomials $P_0 = 1, P_1, \ldots, P_s \in \mathbb{Z}[X]$ with $\deg(P_i) < \deg(P_j)$ for $i < j$ and elements $k_0, \ldots, k_{s-1}, k_s = 0 \in \mathbb{N}_0$ with $k_i > k_j$ for $i < j$ such that

$$\mathrm{Ann}_x = (2^{k_0} Q_x,\ 2^{k_1} P_1 Q_x,\ \ldots,\ 2^{k_{s-1}} P_{s-1} Q_x,\ P_s Q_x).$$

Question 1. From the above example we can see that we must not always have $\deg(P_i) = \deg(P_{i-1}) + 1$ for $i = 1, \ldots, s$. But it can easily be shown that we always have $\deg(P_i) \leq \deg(P_{i-1}) + 2$. Is it possible to find a criterion which indicates whether the difference between $\deg(P_i)$ and $\deg(P_{i-1})$ is 1 or 2? △

Question 2. The example also shows that we must not always have $k_{i-1} = k_i + 1$. Does there exist a criterion to decide how big the differences between the k_i can be? △

Question 3. All the generators in the example are integer multiples of products of linear factors. Is it always possible to choose the P_i as products of linear factors? Can the P_i be chosen such that P_{i-1} divides P_i for $i = 1, \ldots, s$? △

References

[Hur89] Hurrelbrink, Jürgen: Annihilating Polynomials for Group Rings and Witt Rings. In: *Canad. Math. Bull.* 32 (4) (1989), pp. 412–416

[KRW72] Knebusch, Manfred ; Rosenberg, Alex ; Ware, Roger: Structure of Witt Rings and Quotients of Abelian Group Rings. In: *Amer. J. Math.* 94 (1) (1972), pp. 119–155

[Lam05] Lam, Tsit-Yuen: *Introduction to Quadratic Forms over Fields*. American Mathematical Society, 2005

[Lew87] Lewis, David W.: Witt Rings as Integral Rings. In: *Invent. Math.* 90 (1987), pp. 631–633

[Pfi95] Pfister, Albrecht: *London Math. Soc. Lecture Note Series*. Vol. 217: *Quadratic Forms with Applications to Algebraic Geometry and Topology*. Cambridge University Press, 1995

[Sch85] Scharlau, Winfried: *Grundlehren der math. Wissenschaften.* Vol. 270: *Quadratic and Hermitian Forms*. Springer-Verlag Berlin Heidelberg New York Tokyo, 1985